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Cardinal interpolation with polysplines on annuli

O. Kounchev[a,*,1], H. Render[b,2]

[a] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 Acad. G. Bonchev St., 1113 Sofia, Bulgaria
[b] Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n., 26004 Logroño, Spain

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Abstract

Cardinal polysplines of order p on annuli are functions in $C^{2p-2}\left(\mathbb{R}^n \setminus \{0\}\right)$ which are piecewise polyharmonic of order p such that $\Delta^{p-1}S$ may have discontinuities on spheres in $\mathbb{R}^n$, centered at the origin and having radii of the form $e^j,\ j \in \mathbb{Z}$. The main result is an interpolation theorem for cardinal polysplines where the data are given by sufficiently smooth functions on the spheres of radius e^j and center 0 obeying a certain growth condition in $|j|$. This result can be considered as an analogue of the famous interpolation theorem of Schoenberg for cardinal splines.

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1. Introduction

Polysplines have been introduced by the first author as a multivariate analog of splines in one variable, see e.g. [9]. In the monograph [10] applications of polysplines to Multiresolution Analysis and Wavelet Analysis in the spirit of the work of Chui (see [5]) have been given. In this

* Corresponding author.
E-mail addresses: kounchev@math.bas.bg (O. Kounchev), render@gmx.de (H. Render).

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paper an interpolation result for cardinal polysplines on annuli (defined below) will be presented which is motivated by the work of Schoenberg on cardinal spline interpolation, see [19].

Let p and n be natural numbers which are fixed throughout the paper and let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{Z}$ the set of all integers. As in [11–13] a function $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a *cardinal polyspline of order p on annuli* if S is $(2p-2)$-times continuously differentiable and the restriction of S to each open annulus

$$A_j := \{x \in \mathbb{R}^n : e^j < |x| < e^{j+1}\}$$

is a polyharmonic function of order p for $j \in \mathbb{Z}$. Recall that a function f defined on an open set U in $\mathbb{R}^n$ is *polyharmonic of order p* if f is $2p$-times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$ where Δ is the Laplace operator and Δ^p its pth iterate. It is well known that a polyharmonic function is real analytic, hence infinitely differentiable. Hence after differentiating a polyspline $(2p-2)$ times one may have discontinuities only on the spheres $e^j \mathbb{S}^{n-1} = \{e^j y : y \in \mathbb{S}^{n-1}\}$ with $j \in \mathbb{Z}$, where

$$\mathbb{S}^{n-1} = \left\{y \in \mathbb{R}^n : |y| = 1\right\}$$

is the unit sphere. So one may see the spheres $e^j \mathbb{S}^{n-1}$, $j \in \mathbb{Z}$, as the multivariate analog of the notion of the knots $j \in \mathbb{Z}$ of a cardinal spline in the univariate case. Later it will become clear why these radii are of the form e^j, $j \in \mathbb{Z}$.

Schoenberg's famous interpolation theorem for cardinal splines of odd degree says that for data given on the knots $j \in \mathbb{Z}$ of polynomial growth in $j \in \mathbb{Z}$ there exists a cardinal spline interpolating the data which is of the same polynomial growth on the real line, see [19, p. 34]. The aim of this paper is to present an analog of Schoenberg's result for polysplines in the following way: the data are given by functions $d_j : e^j \mathbb{S}^{n-1} \to \mathbb{C}$ for $j \in \mathbb{Z}$ and we want to find a polyspline $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ which interpolates the data, i.e. that

$$S(y) = d_j(y) \text{ for all } y \in e^j \mathbb{S}^{n-1} \text{ and } j \in \mathbb{Z} \tag{1}$$

and which has a similar growth as the data. Clearly we have to assume that the data functions d_j are at least $(2p-2)$ times continuously differentiable. It turns out that the results are naturally formulated in the context of the Sobolev spaces $H^{s,1}(\mathbb{S}^{n-1})$ for appropriate $s > 0$, for details see Section 6.

Our main result states the following: Let $\gamma \geqslant 0$ be fixed; for $s = s_{p,n} = 2(p-1) + (n/2) - 1$ and $f_j \in H^{s,1}(\mathbb{S}^{n-1})$, $j \in \mathbb{Z}$, define functions $d_j : e^j \mathbb{S}^{n-1} \to \mathbb{C}$ by $d_j(e^j \theta) = f_j(\theta)$ for $\theta \in \mathbb{S}^{n-1}$. Assume that the data functions obey the growth condition

$$\left\| f_j \right\|_s \leqslant C |\log e^j|^\gamma \quad \text{for all } j \in \mathbb{Z}.$$

Then there exists a polyspline S of order p interpolating the data functions d_j (i.e. (1)) and satisfying the estimate

$$|S(x)| \leqslant D\, |\log |x||^\gamma \quad \text{for all } x \in \mathbb{R}^n.$$

In order to explain the construction of S recall that a function $u : \mathbb{R} \to \mathbb{R}$ is a *cardinal L-spline* (here L stands for a linear differential operator with constant coefficients of degree $N+1$) if u is $(N-1)$-times continuously differentiable and if for every $l \in \mathbb{Z}$ there exists an infinitely differentiable function $f_l : \mathbb{R} \to \mathbb{C}$ with $Lf = 0$ such that $u(t) = f_l(t)$ for all $t \in (l, (l+1))$. The essence of our construction involves writing the Laplacian in spherical coordinates, expanding the polyspline S in a series of spherical harmonics, and, using the Micchelli theory of cardinal L-splines, glueing the radial part together to get S; roughly speaking, this means that a polyspline can be written in the form

$$S(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} S_{k,l}(\log|x|)\, Y_{k,l}\left(\frac{x}{|x|}\right),$$

where $Y_{k,l}$, $k = 0, 1, \ldots, l = 1, \ldots, a_k$, is a basis for the set of all spherical harmonics and the coefficients $S_{k,l}$ are L-splines with respect to the linear differential operator $M_{\Lambda(k)}$ defined in (3). In order to achieve convergence of the sum one needs precise estimates for the *fundamental L-splines* taking into account their dependence on the parameter k.

The paper is structured as follows: Section 2 gives some basic facts about polysplines and spherical harmonics in order to clarify the connection between polysplines and L-splines. In Section 3 we give a brief account of the theory of Micchelli who has generalized in [16,17] the results of Schoenberg on polynomial splines to the setting of L-splines.

In Section 4 we discuss asymptotic estimates of the Euler–Frobenius function (defined in Section 3) depending on the parameter $k \in \mathbb{N}_0$. In Section 5, we use these asymptotics to obtain uniform estimates of fundamental L-splines containing the parameter k. Section 6 contains our main result. Uniqueness of the interpolation splines will be shown in the last section. In the references [4] and [14] the reader will find recent developments of "interpolation polysplines on strips", where the interpolation data lie on parallel hyperplanes.

2. Spherical harmonics and polysplines

Each $x \in \mathbb{R}^n$ will be written in spherical coordinates $x = r\theta$ with $r \geqslant 0$ and $\theta \in \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Recall that a function $Y : \mathbb{S}^{n-1} \to \mathbb{C}$ is a *spherical harmonic* of degree $k \in \mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P(x)$ of degree k such that $P(\theta) = Y(\theta)$ for all $\theta \in \mathbb{S}^{n-1}$. By a_k we will denote the dimension of the vector space $\mathcal{H}_k$ of all spherical harmonics of degree exactly k. By $Y_{k,l}(\theta)$, $l = 1, \ldots, a_k$ we will denote an orthonormal basis of the space $\mathcal{H}_k$ endowed with the scalar product

$$\int_{\mathbb{S}^{n-1}} f(\theta)\overline{g(\theta)}\, d\theta.$$

For the reader not familiar with spherical harmonics, it might be useful to consider the two-dimensional case: identify $\mathbb{S}^1$ with $[0, 2\pi)$ and choose as a basis $Y_0 = \frac{1}{\sqrt{2\pi}}$ and

$$Y_{k,1}(t) = \frac{1}{\sqrt{\pi}} \cos kt \text{ and } Y_{k,2}(t) = \frac{1}{\sqrt{\pi}} \sin kt.$$

For a detailed account we refer to [23] or [2].

Let $R_1 < R_2$ be positive real numbers and let (R_1, R_2) be the open interval $\{r \in \mathbb{R} : R_1 < r < R_2\}$. Assume that $u : (R_1, R_2) \to \mathbb{C}$ be infinitely differentiable and $Y_k \in \mathcal{H}_k$. Then it is well known (see e.g. [10, p. 152]) that $\Delta(u(r)Y_k(\theta)) = Y_k(\theta)L_{(k)}u(r)$ where

$$L_{(k)} = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{k(k+n-2)}{r^2}. \tag{2}$$

By iteration we have $\Delta^p u = Y_k(\theta) \cdot [L_{(k)}]^p u(r)$. Thus the function $u(r, \theta) = u(r)Y_k(\theta)$ is polyharmonic of order p if and only if $[L_{(k)}]^p u(r) = 0$ for all $r \in (R_1, R_2)$.

Let us put for convenience

$$\Lambda_+(k) := \{k, k+2, \ldots, k+2p-2\},$$
$$\Lambda_-(k) := \{-k-n+2, -k-n+4, \ldots, -k-n+2p\}.$$

The space of solutions of the equation $L^p_{(k)} f(r) = 0$ which are C^∞ for $r > 0$ is generated by a simple basis: for $j \in \Lambda_+(k) \cup \Lambda_-(k)$ the function r^j is clearly a solution, while for $j \in \Lambda_+(k) \cap \Lambda_-(k)$ we obtain a second solution $r^j \log r$. It will be convenient to make a transform $v = \log r$. Then a solution of the form r^j will be transformed to e^{jv} and a solution of the form $r^j \log r$ is transformed to ve^{jv}. We see immediately that all solutions to the equation $L^p_{(k)} f(r) = 0$ are transformed to solutions of the equation $M_{\Lambda(k)} g(v) = 0$ where $M_{\Lambda(k)}$ is the *constant coefficient* linear differential operator defined by

$$M_{\Lambda(k)} := \prod_{\lambda \in \Lambda_+(k)} \left(\frac{d}{dv} - \lambda \right) \prod_{\lambda \in \Lambda_-(k)} \left(\frac{d}{dv} - \lambda \right). \tag{3}$$

Later we shall also use the notation

$$\Lambda(k) = (k, \ldots, k+2p-2, -k-n+2, \ldots, -k-n+2p), \tag{4}$$

which is a vector taking all values from $\Lambda_+(k)$ and $\Lambda_-(k)$ (including multiplicities). From this we have immediately

Proposition 1. *Let N be a natural number and suppose that $S_{k,l} : \mathbb{R} \to \mathbb{C}$ are cardinal L-splines with respect to the differential operator $M_{\Lambda(k)}$ for $k = 0, \ldots, N$, $l = 1, \ldots, a_k$. Then the function $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ defined for $x = r\theta$ with $r > 0$ and $\theta \in \mathbb{S}^{n-1}$ by*

$$S(r\theta) = \sum_{k=0}^{N} \sum_{l=1}^{a_k} S_{k,l}(\log r) Y_{k,l}(\theta)$$

is a cardinal polyspline of order p.

It might be a temptation to say that cardinal polysplines are just the functions of the form

$$S(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} S_{k,l}(\log r)\, Y_{k,l}(\theta), \tag{5}$$

where $S_{k,l}$ are L-splines with respect to $M_{\Lambda(k)}$; however, one has to be careful since the convergence of the sum has to be justified and the differentiability of the function S defined in (5) up to the order $2p-2$ is not a consequence of the absolute convergence of the sum.

On the other hand, we mention the following result in [12] which will be used in the last section to prove uniqueness of interpolation with polysplines.

Theorem 2. *Let $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be a cardinal polyspline of order p. Then the function $S_{k,l} : \mathbb{R} \to \mathbb{C}$ defined by*

$$S_{k,l}(v) := \int_{\mathbb{S}^{n-1}} S(e^v \theta) Y_{k,l}(\theta)\, d\theta \tag{6}$$

is a cardinal L-spline with respect to $M_{\Lambda(k)}$ for $k \in \mathbb{N}_0$, $l = 1, \ldots, a_k$.

3. Cardinal L-splines

The previous section has shown that polysplines are intimately related to a sequence of L-splines given by the Fourier coefficients of the polysplines.

Micchelli has worked out in [16,17] a theory of cardinal L-splines with respect to a linear differential operator L (of order $N+1$) with constant coefficients. As in [16] $\Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_{N+1})$ denotes an (unordered) vector with repetitions according to the multiplicities with real coefficients λ_j, $j = 1, \ldots, N+1$. Then L defined by

$$L := \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j \right)$$

is a linear differential operator of order $N+1$. Let us define the polynomial q_Λ as

$$q_\Lambda(z) := \prod_{j=1}^{N+1} (z - \lambda_j) \tag{7}$$

and $e^\Lambda = \{e^{\lambda_j} : j = 1, \ldots, N+1\}$. In the theory of cardinal L-splines the function $A_\Lambda : \mathbb{R} \times (\mathbb{C} \setminus e^\Lambda) \to \mathbb{C}$ (cf. [17, p. 223]) defined by

$$A_\Lambda(x, \lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{q_\Lambda(z)} \frac{e^{xz}}{e^z - \lambda} \, dz \tag{8}$$

is of fundamental importance. Here Γ is a closed simple curve in the complex plane surrounding all λ_j, $j = 1, \ldots, N+1$ and having the zeros of the function $e^z - \lambda$ in the exterior of Γ. The *Euler–Frobenius function* is defined by

$$\Pi_\Lambda(x, \lambda) := A_\Lambda(x, \lambda) \cdot \prod_{j=1}^{N+1} (e^{\lambda_j} - \lambda). \tag{9}$$

For $x = 0$ it is a polynomial of degree at most N in the variable λ (Corollary 2.1 in [17]) and $\Pi_\Lambda(0, \lambda)$ is called the *Euler–Frobenius polynomial*. Next we recall the definition of the so-called basis spline which will be denoted by Q_Λ: Define the function $s_\Lambda(\lambda) := \prod_{j=1}^{N+1} (e^{-\lambda_j} - \lambda)$ and let s_j, $j = 0, \ldots, N+1$ be the coefficients of $s_\Lambda(\lambda)$, i.e. $s_\Lambda(\lambda) = \sum_{j=0}^{N+1} s_j \lambda^j$. Due to the choice of the real number s_j it is straightforward to prove that the following cardinal L-spline has support in the interval $[0, N+1]$, namely

$$Q_\Lambda(x) := \sum_{j=0}^{N+1} s_j \cdot A_\Lambda(x - j, 0) \cdot 1_{[0,\infty)}(x). \tag{10}$$

The following fundamental formula relates the Euler–Frobenius function with the basis–spline (cf. [17, p. 221 and 222]) for $0 \leqslant x \leqslant 1$,

$$R_\Lambda^x(\lambda) := \sum_{j=0}^{N} \lambda^{N-j} Q_\Lambda(x + j) = \frac{(-1)^N}{e^{(\lambda_1 + \cdots + \lambda_{N+1})}} \cdot \Pi_\Lambda(x, \lambda). \tag{11}$$

3.1. The fundamental L-spline

Let us now consider the interpolation problem for cardinal L-splines. A cardinal L-spline L_Λ is called *fundamental L-spline* if $L_\Lambda(0) = 1$ and $L_\Lambda(j) = 0$ for all $j \in \mathbb{Z}$, $j \neq 0$ and if it decays exponentially, i.e. if there exist two constants $A, B > 0$ such that

$$|L_\Lambda(x)| \leqslant Ae^{-B|x|} \quad \text{for all } x \in \mathbb{R}. \tag{12}$$

We cite the following result from [17, Corollary 2.3].

Theorem 3. *If $A_\Lambda(0, -1) \neq 0$ then there exists a unique fundamental L-spline.*

We now recall from [20, p. 271] the construction of the fundamental spline L_Λ since we need a detailed knowledge of the constants A and B in the estimate (12). Define

$$P_\Lambda(\lambda) := R_\Lambda^0\left(\frac{1}{\lambda}\right)\lambda^N = \sum_{j=0}^{N} \lambda^j Q_\Lambda(j). \tag{13}$$

The following result in [17, Corollary 2.3] shows that P_Λ has no zeros on the unit circle.

Proposition 4. *The function $1/P_\Lambda(\lambda)$ is holomorphic in a neighborhood of the unit circle if and only if $A_\Lambda(0, -1) \neq 0$.*

Assume now that the function $\lambda \to 1/P_\Lambda(\lambda)$ is holomorphic on the annulus $\{R_1 < |\lambda| < R_2\}$ (where $R_1 < 1 < R_2$), and consider its Laurent series

$$\frac{1}{P_\Lambda(\lambda)} = \sum_{j=-\infty}^{\infty} \omega_j \lambda^j.$$

According to [20, p. 271] the fundamental L-spline L_Λ is given by

$$L_\Lambda(x) := \sum_{j=-\infty}^{\infty} \omega_j Q_\Lambda(x - j). \tag{14}$$

The series in (14) converges absolutely and locally uniformly. The estimate in the next proposition is straightforward using the Cauchy estimates for the coefficients of a Laurent series. The somewhat technical proof is omitted.

Proposition 5. *Let $\Lambda = (\lambda_1, \ldots, \lambda_{N+1})$. Suppose that $1/P_\Lambda(\lambda)$ is holomorphic on the annulus $\{R_1 < |\lambda| < R_2\}$ with $R_1 < 1 < R_2$. Let $\rho > 0$ with $R_1 < \rho < 1 < \frac{1}{\rho} < R_2$ and put $\varepsilon = -\log \rho > 0$. Then there exists a constant $G(\rho)$ depending only on ρ and N such that*

$$|L_\Lambda(x)| \leqslant G(\rho) \max_{y \in (0, N+1)} |Q_\Lambda(y)| \cdot \max_{\rho \leqslant |\lambda| \leqslant 1/\rho} \frac{1}{|P_\Lambda(\lambda)|} \cdot e^{-\varepsilon|x|}.$$

We mention that the same proof yields the inequality

$$\left|\frac{d^m}{dx^m} L_\Lambda(x)\right| \leqslant G(\rho) \max_{y \in (0, N+1)} \left|\frac{d^m}{dy^m} Q_\Lambda(y)\right| \cdot \max_{\rho \leqslant |\lambda| \leqslant 1/\rho} \frac{1}{|P_\Lambda(\lambda)|} \cdot e^{-\varepsilon|x|} \tag{15}$$

for each $m = 0, \ldots, N - 1$.

3.2. *Estimate of* max Q_Λ

In the following we want to give an estimate of the basis spline Q_Λ and its derivatives, i.e. we want to estimate $\left|\frac{d^m}{dx^m} Q_\Lambda(x)\right|$ where m satisfies $0 \leqslant m \leqslant N-1$. For this we define for given $\Lambda = (\lambda_1, \ldots, \lambda_{N+1})$ the number

$$M_\Lambda := \max\{|\lambda_1|, \ldots, |\lambda_{N+1}|\}$$

and for $M_\Lambda \neq 0$ we put

$$B_\Lambda(m) := \sum_{k=0}^{m} M_\Lambda^{-k} \max_{0 \leqslant x \leqslant 1} |A_{(\lambda_1,\ldots,\lambda_{N+1-k})}(x,1)|. \tag{16}$$

Note that $B_\Lambda(0) = \max_{0\leqslant x\leqslant 1} |A_{(\lambda_1,\ldots,\lambda_{N+1})}(x,1)|$ and $B_\Lambda(m) \leqslant B_\Lambda(m+1)$.

Recall that $r_\Lambda(\lambda) = \prod_{j=1}^{N+1}(e^{\lambda_j} - \lambda)$.

Theorem 6. *Let $N \in \mathbb{N}_0$ and $\delta > 0$ be given. Then for every $0 \leqslant m \leqslant N-1$ there exists a constant $C_m > 0$, depending only on N and δ, such that for all $\Lambda = (\lambda_1, \ldots, \lambda_{N+1})$ with the property that $|e^{\lambda_j} - 1| \geqslant \delta$ for all $j = 1, \ldots, N+1$, the following inequality*:

$$\left|\frac{d^m}{dx^m} Q_\Lambda(x)\right| \leqslant C_m e^{-(\lambda_1+\cdots+\lambda_{N+1})} M_\Lambda^m \cdot B_\Lambda(m) \cdot |r_\Lambda(1)| \tag{17}$$

holds for all $x \in \mathbb{R}$.

Proof. Let us prove the claim at first for the case $m = 0$: The basis spline Q_Λ is non-negative and it has support in $[0, N+1]$; for $y \in [0, N+1]$ we can find $j \in \{0, 1, \ldots, M\}$ and $x \in [0,1]$ with $y = x + j$. Clearly

$$Q_\Lambda(y) \leqslant \sum_{j=0}^{N} Q_\Lambda(x+j).$$

Taking $\lambda = 1$ in formula (11), one obtains that

$$Q_\Lambda(y) \leqslant \frac{|\Pi_\Lambda(x,1)|}{e^{(\lambda_1+\cdots+\lambda_{N+1})}} = \frac{1}{e^{(\lambda_1+\cdots+\lambda_{N+1})}} |A_\Lambda(x,1) \cdot r_\Lambda(1)|. \tag{18}$$

Hence the claim is true for $m = 0$ where $C_0 = 1$.

We proceed by induction over $m = 0, \ldots, N-1$ and assume that the statement is true for $m \leqslant N-1$. If $m = N-1$ we are done, so assume that $m < N-1$. We apply the induction hypothesis to $\Lambda = (\lambda_1, \ldots, \lambda_{N+1})$ and $\Lambda_2 = (\lambda_1, \ldots, \lambda_N)$ (note that $m \leqslant N-2$), hence for all $x \in \mathbb{R}$

$$\left|\frac{d^m}{dx^m} Q_\Lambda(x)\right| \leqslant C_1 e^{-(\lambda_1+\cdots+\lambda_{N+1})} M_\Lambda^m \cdot B_\Lambda(m) \cdot |r_\Lambda(1)|,$$

$$\left|\frac{d^m}{dx^m} Q_{\Lambda_2}(x)\right| \leqslant C_2 e^{-(\lambda_1+\cdots+\lambda_N)} M_{\Lambda_2}^m \cdot B_{\Lambda_2}(m) \cdot \left|r_{\Lambda_2}(1)\right|.$$

In [7, p. 119] or [10, Part II] one can find the formula

$$\frac{d}{dx}Q_{(\lambda_1,\ldots,\lambda_{N+1})}(x) = \lambda_{N+1}Q_{(\lambda_1,\ldots,\lambda_{N+1})}(x) + e^{-\lambda_{N+1}}Q_{(\lambda_1,\ldots,\lambda_N)}(x) + Q_{(\lambda_1,\ldots,\lambda_N)}(x-1). \tag{19}$$

Differentiating the last equation m times yields

$$\frac{d^{m+1}}{dx^{m+1}}Q_\Lambda(x) = \lambda_{N+1}\frac{d^m}{dx^m}Q_{(\lambda_1,\ldots,\lambda_{N+1})}(x) + e^{-\lambda_{N+1}}\frac{d^m}{dx^m}Q_{(\lambda_1,\ldots,\lambda_N)}(x) + \frac{d^m}{dx^m}Q_{(\lambda_1,\ldots,\lambda_N)}(x-1).$$

The triangle inequality and our induction hypothesis show that

$$\left|\frac{d^{m+1}}{dx^{m+1}}Q_\Lambda(x)\right| \leqslant |\lambda_{N+1}|\, C_1 e^{-(\lambda_1+\cdots+\lambda_{N+1})} M_\Lambda^m \cdot B_\Lambda(m)\cdot |r_\Lambda(1)| + (e^{-\lambda_{N+1}}+1)C_2 e^{-(\lambda_1+\cdots+\lambda_N)} M_{\Lambda_2}^m \cdot B_{\Lambda_2}(m)\cdot |r_{\Lambda_2}(1)|.$$

Now $r_{(\lambda_1,\ldots,\lambda_{N+1})}(1) = (e^{\lambda_{N+1}}-1)r_{(\lambda_1,\ldots,\lambda_N)}(1)$ and $|\lambda_{N+1}| \leqslant M_\Lambda$, and $M_{\Lambda_2}^m \leqslant M_\Lambda^m$. Thus

$$\left|\frac{d^{m+1}}{dx^{m+1}}Q_\Lambda(x)\right| \leqslant e^{-(\lambda_1+\cdots+\lambda_{N+1})}\,|r_\Lambda(1)|\cdot M_\Lambda^{m+1}\cdot C_\Lambda,$$

where

$$C_\Lambda = \left(C_1 B_\Lambda(m) + C_2\frac{1}{M_\Lambda}B_{\Lambda_2}(m)\,\frac{(e^{-\lambda_{N+1}}+1)e^{\lambda_{N+1}}}{\left|e^{\lambda_{N+1}}-1\right|}\right).$$

Further we have the trivial estimate $B_\Lambda(m) \leqslant B_\Lambda(m+1)$ and

$$B_{\Lambda_2}(m) = \sum_{k=1}^{m+1}\max_{0\leqslant x\leqslant 1}\left|M_\Lambda^{-(k-1)}A_{(\lambda_1,\ldots,\lambda_{N+1-k})}(x,1)\right| \leqslant M_\Lambda B_\Lambda(m+1).$$

The function $x \longmapsto |(x+1)(x-1)^{-1}|$ is bounded on $\mathbb{R}\setminus[1-\delta, 1+\delta]$. Since $|e^{\lambda_j}-1| \geqslant \delta$ for all $j = 1,\ldots,N+1$, we infer $C_\Lambda \leqslant C_3 B_\Lambda(m+1)$ where C_3 depends only on N and δ. The proof is complete. □

3.3. Symmetry properties

Let $\Lambda = (\lambda_1,\ldots,\lambda_{N+1})$ and define $-\Lambda = (-\lambda_1,\ldots,-\lambda_{N+1})$. For all $x \in \mathbb{R}$ and $\lambda \notin e^\Lambda \cup e^{-\Lambda}\cup\{0\}$ the following identity (see [17, p. 213]):

$$A_\Lambda\left(1-x, \frac{1}{\lambda}\right) = (-1)^{N+1}\lambda \cdot A_{-\Lambda}(x,\lambda) \tag{20}$$

follows by a direct computation. As in [11] we call Λ *nearly symmetric* if there exists $c \in \mathbb{R}$ and a permutation π of the set $\{1,\ldots,N+1\}$ such that $-\lambda_j = c + \lambda_{\pi(j)}$ for $j = 1,\ldots,N+1$, or shortly $-\Lambda = c + \Lambda$. In the case $c = 0$ we call Λ *symmetric*. Note that for $j \in \{1,\ldots,N+1\}$

with $\pi(j) = j$ one obtains that $-c = \lambda_j + \lambda_{\pi(j)} = 2\lambda_j$ and therefore $\lambda_j = -\frac{1}{2}c$. It follows that

$$\lambda_1 + \cdots + \lambda_{N+1} = -\tfrac{1}{2}(N+1)\,c \tag{21}$$

since $\lambda_j + \lambda_{\pi(j)} = -c$ for $j = 1, \ldots, N+1$. A simple computation shows that for all $x \in \mathbb{R}$ and $\lambda \notin e^{\Lambda} \cup e^{-\Lambda} \cup \{0\}$

$$A_{-\Lambda}(x, \lambda) = e^{(x-1)c} A_{\Lambda}(x, \lambda e^{-c}). \tag{22}$$

Combining Equation (20) and (22) one obtains

Proposition 7. *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$. For all $\lambda \notin e^{\Lambda} \cup e^{-\Lambda} \cup \{0\}$ and all $x \in \mathbb{R}$ the following equality*:

$$A_{\Lambda}\left(1 - x, \frac{1}{\lambda}\right) = (-1)^{N+1} \lambda e^{(x-1)c} A_{\Lambda}(x, \lambda e^{-c}) \tag{23}$$

holds.

Similar computations lead to the following result (cf. Proposition 7 in [11]):

Proposition 8. *Let Λ be nearly symmetric with respect to $c \in \mathbb{R}$. Then the polynomial $P_{\Lambda}(\lambda)$ defined in* (13) *is given by*

$$P_{\Lambda}(\lambda) = (-1)^N \lambda e^{Nc} \cdot \Pi_{\Lambda}(0, \lambda e^{-c}). \tag{24}$$

4. Estimate of the function $A_{\Lambda}(x, \lambda)$

In this section we will give an estimate of the asymptotic behavior of the function $A_{\Lambda(k)}(x, \lambda)$ for $k \to \infty$ and $0 \leqslant x \leqslant 1$. This estimate will be used to prove the existence of an interpolation polyspline for the case that $\Lambda = \Lambda(k)$ is of the form (4).

Assume that for each $k \in \mathbb{N}_0$ the vector $\Lambda = \Lambda(k) = \{\lambda_1(k), \ldots, \lambda_{N+1}(k)\}$ is of the following form: there exists $r \in \{1, \ldots, N+1\}$ (independent of $k \in \mathbb{N}_0$), pairwise different real numbers $C_1, \ldots, C_r$, and pairwise different numbers $C_{r+1.}, \ldots, C_{N+1}$, such that for all $k \in \mathbb{N}_0$ we have the equalities

$$\lambda_j = \lambda_j(k) = \begin{cases} -k + C_j & \text{for } j = 1, \ldots, r, \\ k + C_j & \text{for } j = r+1, \ldots, N+1. \end{cases} \tag{25}$$

Then for large k all $\lambda_j(k)$ are pairwise different for $j = 1, \ldots, N+1$, consequently

$$A_{\Lambda(k)}(x, \lambda) = \sum_{j=1}^{N+1} \frac{1}{q'_{\Lambda(k)}\left(\lambda_j(k)\right)} \frac{e^{\lambda_j(k)x}}{e^{\lambda_j(k)} - \lambda}, \tag{26}$$

where $q'_{\Lambda(k)}$ is the derivative of $q_{\Lambda(k)}$. Let us split $A_{\Lambda(k)}(x, \lambda)$ into a sum of two functions

$$c_k(x, \lambda) = \sum_{j=1}^{r} \frac{1}{q'_{\Lambda}\left(\lambda_j(k)\right)} \frac{e^{\lambda_j(k)x}}{e^{\lambda_j} - \lambda},$$

$$d_k(x, \lambda) = \sum_{j=r+1}^{N+1} \frac{1}{q'_{\Lambda}\left(\lambda_j(k)\right)} \frac{e^{\lambda_j(x)x}}{e^{\lambda_j} - \lambda}.$$

Let K be a compact subset of the complex plane such that $0 \notin K$ and let δ be a positive number. Then it is easy to see that the sequence $(d_k(x, \lambda))_{k \in \mathbb{N}_0}$ with $\lambda \in K$ and $0 \leqslant x \leqslant 1 - \delta$ is of *uniform exponential decay* in the following sense: there exists a polynomial P and $\varepsilon > 0$ such that $|d_k(x, \lambda)| \leqslant |P(k)| \cdot e^{-\varepsilon \cdot k}$ for all $k \in \mathbb{N}_0$, all $\lambda \in K$, and all $0 \leqslant x \leqslant 1 - \delta$.

Let us define

$$b_k(x) = \sum_{j=1}^{r} \frac{e^{\lambda_j(k)x}}{q'_\Lambda\left(\lambda_j(k)\right)}.$$

The following simple result tells us that the asymptotic of $\lambda A_{\Lambda(k)}(x, \lambda)$ for $k \to \infty$ is the same as of $b_k(x)$.

Proposition 9. *Define* $E(k, \lambda) := \prod_{l=1}^{r} (e^{\lambda_l(k)} - \lambda)$ *and let* K *be a compact subset of the complex plane not containing* 0 *and let* $0 < \delta < 1$. *Then we can write*

$$\lambda A_\Lambda(x, \lambda) = \frac{(-\lambda)^r}{E(k, \lambda)} b_k(x) + \lambda f_k(x, \lambda), \tag{27}$$

where $f_k(x, \lambda)$ *is of uniform exponential decay on* $[0, 1 - \delta]$ *and* $E(k, \lambda)$ *converges uniformly on* K *to* $(-\lambda)^r \neq 0$.

Proof. Define $E_j(k, \lambda) := \prod_{l=1, l \neq j}^{r} \left(e^{\lambda_l(k)} - \lambda\right)$. Then $E_j(k, \lambda)$ is a sum of sequences of uniform exponential decay and the constant $(-\lambda)^{r-1}$. It is easy to see that

$$e_k(x, \lambda) := \left(\sum_{j=1}^{r} \frac{e^{\lambda_j(k)x}}{q'_\Lambda\left(\lambda_j(k)\right)} E_j(k, \lambda) \right) - (-\lambda)^{r-1} b_k(x)$$

is of uniform exponential decay. Thus

$$f_k(x, \lambda) := \frac{e_k(x, \lambda)}{E(k, \lambda)} + d_k(x, \lambda) = A_\Lambda(x, \lambda) - \frac{(-\lambda)^{r-1}}{E(k, l)} b_k(x) \tag{28}$$

is of uniform exponential decay. □

Theorem 10. *Let* $\Lambda(k)$ *be as in* (25) *and let* K *be a compact subset of the complex plane with* $0 \notin K$. *Then for each* $\delta > 0$ *there exists a constant* $D > 0$ *and a natural number* k_0 *such that for all* $k \geqslant k_0$, *all* $\lambda \in K$, *and all* $0 \leqslant x \leqslant 1 - \delta$ *the following estimate*:

$$\left| A_{\Lambda(k)}(x, \lambda) \right| \leqslant D \frac{1}{k^N} \tag{29}$$

holds. If there exists $c \in \mathbb{R}$ *such that* $\Lambda(k)$ *is nearly symmetric with respect to* c *for all* $k \geqslant k_0$ *then the inequality is valid for all* $0 \leqslant x \leqslant 1$.

Proof. We may assume that K is disjoint with $e^{\Lambda(k)}$ for large k. Let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ and define $\Gamma_k(t) := -k + k\gamma(t)$. Let $k_0 \in \mathbb{N}_0$ be so large that $\left|C_j\right| < \frac{1}{2} k_0$ for all $j = 1, \ldots, N + 1$. Then for all $k \geqslant k_0$ the curve Γ_k surrounds $\lambda_1, \ldots, \lambda_r$ but not $\lambda_{r+1}, \ldots, \lambda_{N+1}$.

By Cauchy's Theorem

$$b_k(x) = \sum_{j=1}^{r} \frac{e^{\lambda_j x}}{q'_\Lambda(\lambda_j)} = \frac{1}{2\pi i}\int_{\Gamma_k} \frac{e^{zx}}{q_\Lambda(z)}\,dz. \tag{30}$$

Note that $|\lambda_j - z| \geqslant k - \frac{1}{2}k_0 \geqslant \frac{1}{2}k$ for all z on the path Γ_k and for all $j = 1, \ldots, N+1$. Clearly $|e^{zx}| \leqslant e^{x\operatorname{Re}(z)}$ (assuming $0 \leqslant x \leqslant 1$) is bounded for $z \in \Gamma_k$. Hence the standard estimate for line integrals gives for a suitable constant $M > 0$ the inequality

$$|b_k(x)| \leqslant M\frac{1}{k^{N+1}}k$$

for all $0 \leqslant x \leqslant 1$ and $k \geqslant k_0$. By (28) we have uniform exponential decay for $(\lambda f_k(x,\lambda))_{k\in\mathbb{N}_0}$, i.e. there exists a polynomial P and $\varepsilon > 0$ such that $|\lambda f_k(x,\lambda)| \leqslant |P(k)| \cdot e^{-\varepsilon\cdot k}$ for all $k \in \mathbb{N}_0$, all $0 \leqslant x \leqslant 1-\delta$, and all $\lambda \in K$. Since $\frac{(-\lambda)^r}{E(k,\lambda)}$ converges uniformly to 1 it follows that for large k

$$|\lambda A_\Lambda(x,\lambda)| \leqslant \left|\frac{(-\lambda)^r}{E(k,\lambda)} b_k(x)\right| + |\lambda f_k(x,\lambda)| \leqslant 2M\frac{1}{k^N} + |P(k)| \cdot e^{-\varepsilon\cdot k}$$

and (29) is proven for $0 \leqslant x \leqslant 1-\delta$.

For the second statement let $K_1 := K \cup \{1/\lambda e^c : \lambda \in K\}$ and let $\delta = \frac{1}{4}$. Then there exists a constant $D > 0$ such that $\left|A_{\Lambda(k)}(x,\mu)\right| \leqslant D\frac{1}{k^N}$ for all $0 \leqslant x \leqslant 1-\delta$ and for all $\mu \in K_1$. Let now $\frac{1}{2} \leqslant y \leqslant 1$ and define $x = 1-y$. By Equation (23), (replace λ by λe^c and x by y and note that $N+1 = 2p$)

$$A_\Lambda(y,\lambda) = \frac{1}{\lambda e^c} e^{-(y-1)c} A_\Lambda\left(1-y, \frac{1}{\lambda e^c}\right) = \frac{1}{\lambda e^c} e^{xc} A_\Lambda\left(x, \frac{1}{\lambda e^c}\right).$$

Hence $|A_\Lambda(y,\lambda)| \leqslant D_2 D\frac{1}{k^N}$ for all $\frac{1}{2} \leqslant y \leqslant 1$ and the proof is complete. □

Theorem 11. *Let $\Lambda(k)$ be as in* (25) *and let K be a compact subset of the complex plane with $0 \notin K$. If $r < N+1$ then there exist constants $C, D > 0$ and a natural number k_0 such that for all $k \geqslant k_0$ and all $\lambda \in K$:*

$$C\frac{1}{k^N} \leqslant \left|A_{\Lambda(k)}(0,\lambda)\right| \leqslant D\frac{1}{k^N}. \tag{31}$$

Further the following inequality holds for all $\lambda \in (-\infty, 0) \cap K$ and all $k \geqslant k_0$;

$$(-1)^{N+r} A_{\Lambda(k)}(0,\lambda) > 0. \tag{32}$$

Proof. Note that by (30)

$$k^N b_k(x) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{-kx(1-\gamma(t))} \cdot \gamma'(t)}{\prod_{j=1}^{r}\left(\gamma(t) - \frac{C_j}{k}\right)\prod_{j=r+1}^{N+1}\left(-2+\gamma(t) - \frac{C_j}{k}\right)}\,dt. \tag{33}$$

Clearly the denominator of the integrand converges to $(\gamma(t))^r (\gamma(t)-2)^{N+1-r}$. For $x = 0$ the nominator is trivially convergent and hence we see that $k^N b_k(0)$ converges to

$$d_r := \frac{1}{2\pi i}\int_\gamma \frac{1}{z^r (z-2)^{N+1-r}}\,dz.$$

Since γ surrounds $z = 0$ but not $z = 2$ this value can be computed by residue theory (see e.g. Proposition 2.4 in [6, p. 113]) and we obtain

$$d_r = \frac{(-1)^{r-1+N}}{(r-1)!} 2^N (N+1-r) \ldots (N-1).$$

It follows that there exist a constant $C > 0$ and an integer k_0 such that $(-1)^{r-1+N} b_k(0) \geqslant C \frac{1}{k^N}$ for all $k \geqslant k_0$:

Assume now that $K \subset (-\infty, 0)$. Since for $k \longrightarrow \infty$ we have $\frac{(-\lambda)^r}{E(k,\lambda)} \longrightarrow 1$ uniformly on K, there exists an integer k_1 such that for all $k \geqslant k_1$ and all $\lambda \in K$:

$$\frac{(-\lambda)^r}{E(k,\lambda)} (-1)^{r-1+N} b_k(0) \geqslant \frac{C}{2} \frac{1}{k^N} > 0.$$

Since the sequence $(\lambda f_k(0,\lambda))_{k \in \mathbb{N}_0}$ is of uniform exponential decay there exists a polynomial P and a number $\varepsilon > 0$ such that $|\lambda f_k(0,\lambda)| \leqslant |P(k)| \cdot e^{-\varepsilon \cdot k}$ for all $k \in \mathbb{N}_0$ and for all $\lambda \in K$. Then by (27) the following inequalities hold:

$$\begin{aligned}(-1)^{r-1+N} \lambda A_\Lambda(0,\lambda) &\geqslant \frac{(-\lambda)^r}{E(k,\lambda)} (-1)^{r-1+N} b_k(0) - |\lambda f_k(0,\lambda)| \\ &\geqslant \frac{C}{2} \frac{1}{k^N} - |P(k)| \cdot e^{-\varepsilon \cdot k} \geqslant \frac{1}{4} \frac{C}{k^N}\end{aligned}$$

for all sufficiently large k and for all $\lambda \in K$. Since the set K contains only negative numbers we obtain estimate (32) for all sufficiently large k.

Now assume that K is a compact subset in the complex plane $\mathbb{C}$. Then similar arguments as above show that for some $k_1 \in \mathbb{N}_0$ the inequality $|\lambda A_\Lambda(0,\lambda)| \geqslant \frac{1}{4} \frac{C}{k^N}$ holds for all $\lambda \in K$, and for all $k \geqslant k_1$. □

5. Uniform estimates of fundamental *L*-splines

In the rest of the paper we will assume that $\Lambda(k)$ is given by (4). We write $\lambda_j(k) = -k + C_j$ for $j = 1, \ldots, p$ with

$$C_1 = 2 - n, C_2 = 4 - n, \ldots, C_p = 2p - n$$

and $\lambda_j(k) = k + C_j$ for $j = p+1, \ldots, 2p$ with

$$C_{p+1} = 0, C_{p+2} = 2, \ldots, C_{2p} = 2p - 2.$$

Hence $N + 1 = 2p$ and clearly $\Lambda(k)$ is *nearly symmetric* with respect to $c = n - 2p$ where $n \in \mathbb{N}_0$ is the dimension of the underlying space $\mathbb{R}^n$.

Theorem 12. *Let $\Lambda(k)$ be as in* (4) *and let K be a compact subset of the complex plane with $0 \notin K$. Then there exist a constant $M > 0$ and an integer k_0 such that $P_{\Lambda(k)}(\lambda) \neq 0$ for all $k \geqslant k_0$ and for all $\lambda \in K$; further for all $k \geqslant k_0$:*

$$C(k) := \max_{x \in [0,1]} Q_{\Lambda(k)}(x) \cdot \max_{\lambda \in K} \frac{1}{\left|P_{\Lambda(k)}(\lambda)\right|} \leqslant M. \tag{34}$$

More generally, for every $m = 0, \ldots, 2p-2$ *there exist a constant* $M_1 > 0$ *and an integer* k_1 *such that for all* $\lambda \in K$ *and for all* $k \geqslant k_1$:

$$C_m(k) := \max_{x\in[0,1]} \left| \frac{d^m}{dx^m} Q_{\Lambda(k)}(x) \right| \cdot \max_{\lambda \in K} \frac{1}{\left| P_{\Lambda(k)}(\lambda) \right|} \leqslant M_1 k^m. \tag{35}$$

Proof. Using $N+1 = 2p$ and $c = n - 2p$ Proposition 8 yields

$$P_{\Lambda(k)}(\lambda) = (-1)\,\lambda e^{Nc} A_{\Lambda(k)}(0, \lambda e^{-c}) \cdot r_{\Lambda(k)}(\lambda e^{-c}), \tag{36}$$

where $r_{\Lambda(k)}(\lambda) = \prod_{j=1}^{2p} (e^{\lambda_j(k)} - \lambda)$. By Theorem 11 applied to the compact set $e^{-c}K := \{e^{-c}\lambda : \lambda \in K\}$ there exists $C > 0$ and $k_0 \in \mathbb{N}_0$ such that $C \leqslant \left| A_{\Lambda(k)}(0, \lambda e^{-c}) \right| \cdot k^{2p-1}$ for all $\lambda \in K$ and for all $k \geqslant k_0$. Thus by (36) $P_{\Lambda(k)}(\lambda) \neq 0$ for all $\lambda \in K$ and for all $k \geqslant k_0$ and the first statement is proven. Furthermore, we have obtained the estimate

$$\frac{1}{\left| P_{\Lambda(k)}(\lambda) \right|} \leqslant \frac{e^{-Nc}}{C|\lambda|} k^{2p-1} \frac{1}{r_{\Lambda(k)}(\lambda e^{-c})}.$$

In order to prove (34) we apply Theorem 6 with $m = 0$, and obtain

$$\left| Q_{\Lambda(k)}(x) \right| \leqslant C e^{-(\lambda_1 + \cdots + \lambda_{N+1})} \max_{0 \leqslant y \leqslant 1} \left| A_{\Lambda(k)}(y, 1) \right| \cdot \left| r_{\Lambda(k)}(1) \right|.$$

Theorem 10 shows that there exists $D_1 > 0$ such that

$$\max_{x\in[0,1]} Q_{\Lambda(k)}(x) \leqslant D_1 e^{p(n-2p)} \frac{1}{k^{2p-1}} \left| r_{\Lambda(k)}(1) \right|.$$

Hence we obtain for a suitable constant D_2 (note that $0 \notin K$) the inequality

$$C(k) \leqslant D_2 \left| r_{\Lambda(k)}(1) \right| \max_{\lambda \in K} \frac{1}{\left| r_{\Lambda(k)}(\lambda e^{-c}) \right|}.$$

The proof is accomplished by the fact that

$$\frac{r_{\Lambda(k)}(1)}{r_{\Lambda(k)}\left(\lambda e^{-c}\right)} = \frac{\prod_{k=1}^{p} \left(e^{-k+C_j} - 1\right) \prod_{k=p+1}^{2p} \left(e^{k+C_j} - 1\right)}{\prod_{k=1}^{p} \left(e^{-k+C_j} - \lambda e^{-c}\right) \prod_{k=p+1}^{2p} \left(e^{k+C_j} - \lambda e^{-c}\right)}$$

converges uniformly for $k \to \infty$ to $\frac{1}{(\lambda e^{-c})^p}$. Estimate (35) follows in the same way using again Theorems 6 and 10. □

For the proof of our main result we need the following proposition which establishes an uniform estimate of the type (12) of all fundamental splines for the operators L generated by the vectors $\Lambda(k)$.

Proposition 13. *For every* $k \in \mathbb{N}_0$ *let* $\Lambda(k)$ *be as in* (4). *Then there exists a fundamental L-spline* $L_{\Lambda(k)}$ *with respect to the operator* $M_{\Lambda(k)}$. *Further there exist constants* $M > 0$ *and* $\varepsilon > 0$ *such that for all* $k \in \mathbb{N}_0$ *and all* $v \in \mathbb{R}$ *the following estimate holds*:

$$\left| L_{\Lambda(k)}(v) \right| \leqslant M e^{-\varepsilon |v|}. \tag{37}$$

Proof. At first we show that $A_{\Lambda(k)}(0,-1) \neq 0$ for all $k \in \mathbb{N}_0$. The integral

$$A_{\Lambda(k)}(0,-1) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{q_{\Lambda(k)}(z)}\frac{1}{e^z+1}\,dz \tag{38}$$

can be computed by residue theory and it reduces to a rational expression which has a non-zero denominator. For simplicity let us consider the case when the constants $\lambda_j(k)$ are pairwise distinct. Then

$$A_{\Lambda(k)}(0,-1) = \sum_{j=1}^{2p} \frac{1}{q'_{\Lambda(k)}\left(\lambda_j(k)\right)}\frac{1}{e^{\lambda_j(k)}+1}.$$

Obviously, $q'_{\Lambda(k)}\left(\lambda_j(k)\right)$ are integers. Let us assume that $A_{\Lambda(k)}(0,-1) = 0$. After multiplying by $\prod_{j=1}^{2p}\left(e^{\lambda_j(k)}+1\right)$ we arrive at an equation of the type

$$\sum_{i=1}^{l} \beta_i e^{\rho_i} = 0,$$

here β_i are non-zero rationals and ρ_i are integers obtained by sums of some of the constants $\lambda_j(k)$. Due to the special form of the constants $\lambda_j(k)$ provided in (4) at least one of the ρ_i is non-zero. Thus we may apply the classical theorem of *Lindemann* on transcendental numbers which states that the above equality is impossible, see e.g. [15, p. 213] or [3, p. 6]. It follows that $A_{\Lambda(k)}(0,-1) \neq 0$.

By Theorem 3 we can find for each $k \in \mathbb{N}_0$ a fundamental L-spline $L_{\Lambda(k)}: \mathbb{R} \to \mathbb{R}$. Hence, there exist constants M_k and ε_k such that for all $v \in \mathbb{R}$ holds

$$\left|L_{\Lambda(k)}(v)\right| \leqslant M_k e^{-\varepsilon_k |v|}.$$

We have to show that the constants M_k can be chosen as a bounded sequence, and similarly that $\varepsilon_k \geqslant \varepsilon$ for all $k \in \mathbb{N}_0$. Let $0 < \rho < 1$ and put $K := \{\lambda \in \mathbb{C} : \rho \leqslant |\lambda| \leqslant 1/\rho\}$. Choose arbitrary ρ^* with $0 < \rho^* < \rho$ and put $T := \{\lambda \in \mathbb{C} : \rho^* \leqslant |\lambda| \leqslant 1/\rho^*\}$. By Theorem 12 applied to the compact set T there exists $k_0 \in \mathbb{N}_0$ such that

$$P_{\Lambda(k)}(\lambda) \neq 0$$

for all $\lambda \in T$ and for all $k \geqslant k_0$. Hence $P_{\Lambda(k)}$ is holomorphic on the open annulus given by the radii $R_1 = \rho^* < 1 < 1/\rho^* = R_2$ for all $k \geqslant k_0$. Again by Theorem 12 applied to the compact set K there exist a constant $M^* > 0$ and a natural number $k_1 \geqslant k_0$ such that

$$C(k) := \max_{x\in[0,1]} Q_{\Lambda(k)}(x) \cdot \max_{\lambda \in K} \frac{1}{\left|P_{\Lambda(k)}(\lambda)\right|} \leqslant M^* \tag{39}$$

for all $k \geqslant k_1$. Apply now Proposition 5 with respect to all sets $\Lambda(k)$ with $k \geqslant k_1$. It follows that there exists a constant $G(\rho)$ (independent of k) such that the fundamental L-splines $L_{\Lambda(k)}$ for $k \geqslant k_1$ can be estimated by

$$\left|L_{\Lambda(k)}(v)\right| \leqslant G(\rho)\, C(k)\, e^{-\varepsilon^*|v|} \leqslant G(\rho)\, M^* e^{-\varepsilon^*|v|},$$

where $\varepsilon^* := -\log \rho$. Finally after putting $M := \max\{M^*, M_0, \ldots, M_{k_1-1}\}$ and $\varepsilon := \min\{\varepsilon^*, \varepsilon_0, \ldots, \varepsilon_{k_1-1}\}$ the proof is complete. □

6. The main result

At first we need some notations: assume that the function $f:\mathbb{S}^{n-1}\to\mathbb{R}$ be square-integrable with respect to the surface measure $d\theta$ on $\mathbb{S}^{n-1}$and define the usual scalar product

$$\langle f, g\rangle_{L^2(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} f(\theta)\overline{g(\theta)}\,d\theta.$$

Recall that $Y_{k,l}(\theta)$, for $k\in\mathbb{N}_0, l=1,\ldots,a_k$ denotes an orthonormal basis of the space $\mathcal{H}_k$ of all spherical harmonics with respect to $d\theta$. For all $k\in\mathbb{N}_0$, and $l=1,\ldots,a_k$ the Fourier–Laplace coefficients of f are given by

$$f_{k,l} := \int_{\mathbb{S}^{n-1}} f(\theta)Y_{k,l}(\theta)\,d\theta.$$

By [23, Corollary 2.3] every square-integrable function f can be expanded into a *Fourier–Laplace series* given by

$$f(\theta) = \sum_{k=0}^{\infty}\sum_{l=1}^{a_k} f_{k,l}\cdot Y_{k,l}(\theta), \tag{40}$$

where convergence is understood in $L_2(\mathbb{S}^{n-1})$ with the norm

$$\|f\|_{L_2(\mathbb{S}^{n-1})} = \sqrt{\langle f, f\rangle_{L_2(\mathbb{S}^{n-1})}}.$$

For every $f\in L_2(\mathbb{S}^{n-1})$ define

$$\|f\|_s := \sum_{k=0}^{\infty}\sum_{l=1}^{a_k} \left|f_{k,l}\right|\cdot(1+k)^s\,. \tag{41}$$

The subspace of all $f\in L_2(\mathbb{S}^{n-1})$ with $\|f\|_s<\infty$ is denoted by $H^{s,1}(\mathbb{S}^{n-1})$, see [1].

By [21], for all $Y_k\in\mathcal{H}_k$ we have the inequality

$$|Y_k(\theta)| \leqslant K k^{(n/2)-1}\,\|Y_k(\theta)\|_{L_2(\mathbb{S}^{n-1})} \quad \text{for } \theta\in\mathbb{S}^{n-1}.$$

Since $\|Y_{k,l}(\theta)\|_{L_2(\mathbb{S}^{n-1})}=1$ we obtain the estimate

$$\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\left|f_{k,l}\right|\left|Y_{k,l}(\theta)\right| \leqslant K\sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\left|f_{k,l}\right|(1+k)^{\frac{n}{2}-1} = K\,\|f\|_{\frac{n}{2}-1}\,. \tag{42}$$

It follows that a function $f\in H^{\frac{n}{2}-1,1}(\mathbb{S}^{n-1})$ possesses an absolutely uniformly convergent Fourier–Laplace series.

Using some standard techniques (see e.g. [8]) one can prove the following criterion:

Proposition 14. *Assume that $f:\mathbb{S}^{n-1}\to\mathbb{R}$ is a $2q$-continuously differentiable function where $2q\geqslant 2(p-1)+2\left[\frac{n}{2}\right]$. Then $f\in H^{s,1}(\mathbb{S}^{n-1})$ for $s=2(p-1)+(n/2)-1$.*

6.1. Construction of fundamental polysplines

As in the one-dimensional case we show at first the existence of "fundamental polysplines" in the following sense:

Definition 15. A fundamental polyspline $L_f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ for the data function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ is the polyspline of order p such that for each $j \in \mathbb{Z}$ the interpolation conditions

$$\begin{aligned} L_f\left(e^j \theta\right) &= 0 && \text{for all } j \neq 0 \text{ and } \theta \in \mathbb{S}^{n-1}, \\ L_f\left(e^j \theta\right) &= f(\theta) && \text{for } j = 0 \text{ and all } \theta \in \mathbb{S}^{n-1} \end{aligned} \tag{43}$$

hold, as well as the following growth condition:

$$\left|L_f(r\theta)\right| \leqslant M e^{-\varepsilon|\log r|} \quad \text{for all } r > 0 \text{ and } \theta \in \mathbb{S}^{n-1}. \tag{44}$$

The next result ensures the existence of fundamental polysplines for a large class of data functions.

Theorem 16. *Let $s = s_{p,n} = 2(p-1) + (n/2) - 1$. Then there exist constants $M > 0$ and $\varepsilon > 0$ with the following property: for each $f \in H^{s,1}(\mathbb{S}^{n-1})$ there exists a polyspline L_f of order p such that* (43) *holds and*

$$\left|\frac{d^m}{dr^m} D^\alpha L_f(r\theta)\right| \leqslant M e^{-\varepsilon|\log r|} \, \|f\|_s \tag{45}$$

for all $m \in \mathbb{N}_0$ and $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1}$ satisfying the condition $m + |\alpha| \leqslant 2p - 2$; here D^α denotes the differential operator

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial \theta_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-1}}}{\partial \theta_{n-1}^{\alpha_{n-1}}}.$$

Proof. Let $L_{\Lambda(k)}$ denote the fundamental cardinal L-spline with respect to the differential operator $M_{\Lambda(k)}$. Now using the Fourier–Laplace series of f we want to define a fundamental polyspline L_f by

$$L_f(r\theta) := \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l} \cdot L_{\Lambda(k)}(\log r) \cdot Y_{k,l}(\theta). \tag{46}$$

The series converges absolutely and uniformly since by (37) and (42) we have the estimate:

$$\left|L_f(r\theta)\right| \leqslant M e^{-\varepsilon|\log r|} \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \left|f_{k,l}\right| \left|Y_{k,l}(\theta)\right| \leqslant K \, \|f\|_{\frac{n}{2}-1}. \tag{47}$$

Furthermore L_f is polyharmonic on each annulus $A(e^j, e^{j+1})$ since each summand $L_{\Lambda(k)}(\log r) \cdot Y_{k,l}(\theta)$ is according to the results in Section 2 polyharmonic of order p and the uniform limit of such functions is again polyharmonic of order p.

Since $L_{\Lambda(k)}(0) = 1$ and $L_{\Lambda(k)}(j) = 0$, for all $j \in \mathbb{Z}$, $j \neq 0$, we conclude that L_f interpolates the given data f, i.e. (43) holds.

We want to prove that the partial derivatives of $\theta \to L_f(r\theta)$ and $r \to L_f(r\theta)$ exist up to the order $2(p-1)$. It suffices to prove the uniform convergence of the series

$$\sum_{k=0}^{\infty}\sum_{l=1}^{a_k} f_{k,l} \cdot \frac{d^m}{dr^m} L_{\Lambda(k)}(\log r) \cdot D^{\alpha} Y_{k,l}(\theta) \tag{48}$$

for $m + |\alpha| \leqslant 2p - 2$. By formula (15), and Theorem 12, there exist constants $C > 0$ and $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ holds

$$\left| \frac{d^m}{dr^m} L_{\Lambda(k)}(\log r) \right| \leqslant C k^m e^{-\varepsilon |\log r|}.$$

By [22], or [21], there exists a constant $K > 0$ independent of k such that for all $Y_k \in \mathcal{H}_k$, and for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \leqslant N := 2(p-1) - m$, the following estimate holds:

$$\left| D^{\alpha} Y_k(\theta) \right| \leqslant K \cdot k^{(n/2)-1+N} \, \| Y_k(\theta) \|_{L_2(\mathbb{S}^{n-1})}.$$

Applying the last inequality to $Y_{k,l}(\theta)$ (note that $\left\| Y_{k,l}(\theta) \right\|_{L_2(\mathbb{S}^{n-1})} = 1$) we obtain that for all $\alpha \in \mathbb{N}_0^{n-1}$ with $|\alpha| \leqslant N := 2(p-1) - m$:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{l=1}^{a_k} \left| f_{k,l} \cdot \frac{d^m}{dr^m} L_{\Lambda(k)}(\log r) \cdot D^{\alpha} Y_{k,l}(\theta) \right| \\
&\quad \leqslant C K e^{-\varepsilon |\log r|} \sum_{k=0}^{\infty}\sum_{l=1}^{a_k} |f_{k,l}| \cdot k^m \cdot k^{(n/2)-1+N} \\
&\quad = C K e^{-\varepsilon |\log r|} \sum_{k=0}^{\infty}\sum_{l=1}^{a_k} \left| f_{k,l} \right| \cdot k^{2(p-1)} \cdot k^{(n/2)-1}.
\end{aligned}$$

Since $\| f \|_{s_{p,n}} < \infty$ we conclude that $L_f(r\theta)$ is differentiable up to the order $2(p-1)$ and (45) holds. □

6.2. *Construction of interpolation polysplines*

Now let us construct interpolation polysplines. Assume that d_j are data functions defined on the spheres $e^j \mathbb{S}^{n-1}$. Then we put $f_j(\theta) := d_j(e^j\theta)$, consequently f_j is a function on the sphere $\mathbb{S}^{n-1}$.

Theorem 17. *Let $\gamma \geqslant 0$ and $s = s_{p,n} = 2(p-1) + (n/2) - 1$ and $f_j \in H^{s,1}(\mathbb{S}^{n-1})$ for $j \in \mathbb{Z}$. Suppose that there exists a constant $C > 0$ such that the inequality*

$$\left\| f_j \right\|_s \leqslant C\,|j|^{\gamma} = C \left| \log e^j \right|^{\gamma} \tag{49}$$

holds for all $j \in \mathbb{Z}$. Then there exists a polyspline $S: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of order p such that

$$S(e^j\theta) = f_j(\theta) = d_j(e^j\theta) \quad \textit{for all } \theta \in \mathbb{S}^{n-1}$$

holds for each $j \in \mathbb{Z}$, and there exists a constant $D > 0$ such that for all $\theta \in \mathbb{S}^{n-1}$ and all $r > 0$:

$$|S(r\theta)| \leqslant D\,|\log r|^{\gamma}.$$

Proof. The following well-known fact can be found, e.g. in [18]: Let $\gamma \geqslant 0$ and $\varepsilon > 0$. Then there exists $D(\varepsilon, \gamma) > 0$ and $R_0 > 0$ such that for all $x \in \mathbb{R}$ with $|x| \geqslant R_0$ the following inequality holds:

$$\sum_{j=-\infty}^{\infty} |j|^{\gamma} e^{-\varepsilon|x-j|} \leqslant D(\varepsilon, \gamma) |x|^{\gamma}. \tag{50}$$

For each f_j we can define a fundamental polyspline L_{f_j} as in Theorem 16. We define the interpolation polyspline by putting

$$S(x) := \sum_{j=-\infty}^{\infty} L_{f_j}\left(xe^{-j}\right).$$

Estimate (45) yields $|L_{f_j}(xe^{-j})| \leqslant Me^{-\varepsilon\left|\log|xe^{-j}|\right|} \| f_j \|_s$, hence by (49) and (50) it follows

$$|S(x)| \leqslant \sum_{j=-\infty}^{\infty} MCe^{-\varepsilon\left|\log|xe^{-j}|\right|} |j|^{\gamma} \leqslant CMD(\varepsilon, \gamma) |\log|x||^{\gamma}.$$

This shows that S is well-defined and since the convergence is locally uniform it is clear that S is continuous on $\mathbb{R}^n \setminus \{0\}$ and polyharmonic on the open annuli $A\left(e^j, e^{j+1}\right)$ for $j \in \mathbb{Z}$.

The differentiability of S up to order $2(p-1)$ follows from similar estimates using inequality (45). Then

$$\sum_{j=-\infty}^{\infty} \left| \frac{d^m}{dr^m} D^{\alpha} L_{f_j}(r\theta e^{-j}) \right| \leqslant \sum_{j=-\infty}^{\infty} Me^{-\varepsilon\left|\log re^{j}\right|} \| f_j \|_s.$$

This ends the proof. □

7. Uniqueness of interpolation polysplines

In this section we will prove uniqueness of interpolation polysplines.

Theorem 18. *Let $\gamma \geqslant 0$. Suppose $S_1, S_2 : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be polysplines of order p such that*

$$|S_i(r\theta)| \leqslant C\left(|\log r|^{\gamma}\right)$$

for $i = 1, 2$. If $S_1\left(e^j\theta\right) = S_2\left(e^j\theta\right)$ for all $j \in \mathbb{Z}$ and for all $\theta \in \mathbb{S}^{n-1}$ then $S_1 \equiv S_2$.

Proof. Let us put $S := S_1 - S_2$. Let $S_{k,l}(\log r)$, with $v = \log r$, be the Fourier–Laplace coefficients of S as defined in (6). According to Theorem 2, $S_{k,l}(v)$ are cardinal L-splines with respect to the linear differential operator $M_{\Lambda(k)}$ and clearly $S_{k,l}(j) = 0$ for all $j \in \mathbb{Z}$. Further, by the assumption of the Theorem we see that for all $v \in \mathbb{R}$ inequality

$$\left|S_{k,l}(v)\right| \leqslant \int_{\mathbb{S}^{n-1}} \left|S(e^v\theta)Y_{k,l}(\theta)\right| d\theta \leqslant C_{k,l} |\log e^v|^{\gamma} = C_{k,l} |v|^{\gamma}$$

holds with some constants $C_{k,l} > 0$. Hence $S_{k,l}$ is a cardinal L-spline of polynomial growth. By the uniqueness for interpolation cardinal L-splines (see [16, p. 204] applied for $\alpha = 0$) we infer that $S_{k,l} \equiv 0$. This implies $S \equiv 0$ and finishes the proof. □

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